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Quadratic Programming Contact Formulation for Elastic Bodies Using Boundary Element Method

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A method for the analysis of contact of deformable bodies based on the boundary element method (BEM) has been presented in this paper. The contact problem is stated in the form of a convex quadratic programming (QP) problem written in terms of the contact tractions on the contact surface. A strategy for the incorporation of the BEM contact analysis into models whose domain may be discretized using the finite element method (FEM) has been investigated. A discussion concerning the merits of the proposed approach is provided and several examples are presented to illustrate the validity of the method.

I. Introduction

THE frictionless contact of deforming elastic bodies is studied in this paper using the boundary element method (BEM). Because of the underlying complexity of the contact problems, only a limited number of analytical solutions, which are restricted to simple geometries and loading conditions, exist in the literature. An overview of the existing analytical solutions may be found in Ref. 1. A numerical discretization of the governing relationships must be employed for the analysis of complex contact configurations. The numerical method used for such discretization in the past has mainly been the finite element method (FEM), which has been employed for obtaining approximations of the solutions to contact variational inequalities.[2] A solution to the variational inequalities may be obtained by a constrained minimization of the governing functionals, such as the total potential energy functional, over an admissible set of functions.[2] The fact that domain-based functionals are easier to model with the FEM and the wide applicability of the method have resulted in significant research on contact problems based on the FEM formulations. The contact constraints in such formulations may be introduced by 1) methods where the kinematic contact constraints are directly imposed in an incremental loading of the structure,[3–7] 2) Lagrangian-based methods,[8–10] and 3) penalty methods.[2,11] Alternative minimum principles defined on the contact boundary also provide effective methods for contact analysis. These formulations may be efficiently discretized using both the FEM and the BEM; for example, see Refs. 12–14. The use of the BEM is advantageous because of the accuracy of the boundary stresses that it provides and the reduced dimensionality of the resulting discretized BEM models compared with those obtained for the domain-based methods.[15] The solution to the contact problems may also be approached directly by satisfying the complementarity conditions on the contact boundary.[16]

Contact problems can be alternatively stated in the form of mathematical programming (MP) problems. Such formulations are derived either from boundary variational formulations or from the direct discretization of the contact complementarity conditions. Since the first publications[17–19] on the subject, considerable research has been performed, mainly in FEM framework.[12,13,20] Recent studies[14,16,21,22] have illustrated the potential and the advantages of the combined BEM and MP approach for the analysis of contact problems. An overview of the research in this area may be found, for example, in Ref. 23.

The present study is directed towards the development of mathematical relations that, combined with MP techniques and the BEM, may be used to solve the problem of frictionless contact between deforming elastic bodies. A minimum principle that is derived from the nonpenetration condition on the contact boundary is used. This relation is expressed, using BEM discretization, in the standard form of the quadratic programming (QP) problems in optimization theory. The conditions of linear, elastic, small deformations are assumed. The proposed method is based on the principle of linear superposition, and it cannot be used for the analysis of contact problems with material and geometrical nonlinearities. A strategy for the incorporation of the BEM contact analysis into models whose domain may be discretized using the FEM has been presented to evaluate the possibility of using the present formulation in existing FEM computer codes. Several numerical examples are provided to illustrate the validity of the present developments.

II. Theoretical Formulation

A. Extended BEM Formulation

A BEM formulation for the contact of elastic bodies against rigid surfaces was presented in Ref. 14 and is extended here to account for the contact of deformable elastic bodies. Consider two elastic bodies occupying their respective volumes Ω^k, $k = 1, 2$, that may come into frictionless contact with each other under the action of external influences as shown in Fig. 1. The governing differential equations and the corresponding boundary conditions may be written as

Equilibrium:

$$(\lambda^k + \mu^k)u^k_{j,ij} + \mu^k u^k_{i,jj} + \Psi^k_i = 0; \qquad x \in \Omega^k, k = 1, 2 \quad (1)$$

Boundary conditions:

$$u^k_i(x) = \bar{u}^k_i(x); \qquad x \in \Gamma^k_u \quad (2)$$

$$t^k_i(x) = \bar{t}^k_i(x); \qquad x \in \Gamma^k_t \quad (3)$$

Compatibility conditions:

$$t^n(x) = t^{1^n}_c(x) = -t^{2^n}_c(x); \qquad x \in \Gamma^k_c \quad (4)$$

Contact conditions:

$$\left.\begin{array}{ll} t^n(x) = 0 & \text{for} \quad \delta^n > 0 \\ t^n(x) > 0 & \text{for} \quad \delta^n = 0 \end{array}\right\}; \qquad x \in \Gamma^k_c \quad (5)$$

$$t^\tau = 0; \qquad x \in \Gamma^k_c \quad (6)$$

$$\delta^n(x) = \epsilon^n(x) + u^{1^n}(x) - u^{2^n}(x) \quad (7)$$

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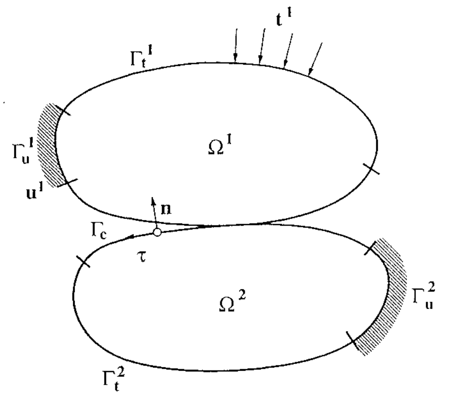

Fig. 1 Elastic bodies in contact.

where the superscript k denotes an elastic body; λ^k and μ^k are the Lame constants; u_i^k and t_i^k denote the displacements and tractions, respectively; Ψ_i^k are the body forces; an overbar denotes prescribed values; Γ_u^k and Γ_t^k are the portions of the surface Γ^k that bounds the volume Ω^k, on which the displacements and the tractions, respectively, are specified; Γ_c^k is the portion of Γ^k that is likely to come into contact with the other body; t^n and u^{kn} denote, respectively, the pressure on the displacement of the elastic body along the normal n to the contact surface t^τ is the pressure on the elastic body along the unit tangent τ to the contact surface Γ_c; ϵ^n is the initial gap or clearance between the elastic bodies along the normal n to Γ_c; and $\Gamma^k = \Gamma_u^k + \Gamma_t^k + \Gamma_c^k$. It is noted that the boundary condition in Eq. (6) results from the fact that the contact between the elastic bodies is frictionless. The actual contact area Γ_c must be fully included in Γ_c^k. This does not pose a restriction since a secure overestimate can be made in most cases. The contact between the two bodies is assumed to occur in a node-to-node fashion such that the pairs of nodes that come in contact are known in advance. The present method is concerned with contact of elastic bodies experiencing small displacements so that the previous requirement does not significantly influence the discretization.

Applying the procedures of the direct BEM formulations, Eq. (1) leads to the matrix relation between boundary displacements and tractions as

$$F^k u^k = G^k t^k + f^k \tag{8}$$

where F^k and G^k are the BEM system matrices; u^k and t^k are the vectors of nodal displacements and tractions, respectively; and f^k is a vector incorporating such volume effects as body forces and thermal loads, etc. The details for obtaining Eq. (8) and for the numerical evaluation of various matrices may be found in standard textbooks, for example, Banerjee and Butterfield.[24] To incorporate the BEM contact analysis into models discretized using the FEM, Eq. (8) may be modified to include the contribution of the FEM region. This is accomplished by converting the FEM region into an equivalent BEM zone.[25] In the following discussion, the index k denoting the body under consideration has been dropped for clarity. Assume that the body Ω is divided in two parts Ω^1 and Ω^2, where the indices 1 and 2 denote the BEM and the FEM discretization regions, respectively. For the BEM region, Eq. (8) may be written in the partitioned form as

$$\begin{bmatrix} F^1 & F_I^1 \end{bmatrix} \begin{Bmatrix} u^1 \\ u_I^1 \end{Bmatrix} = \begin{bmatrix} G^1 & G_I^1 \end{bmatrix} \begin{Bmatrix} t^1 \\ t_I^1 \end{Bmatrix} \tag{9}$$

where the subscript I denotes the interface between the FEM and the BEM regions. In the FEM region, using the interpolation matrices M^2 and M_I^2 to relate the boundary tractions to nodal forces, one may write the standard stiffness relations as

$$\begin{bmatrix} K^2 & K_I^2 \end{bmatrix} \begin{Bmatrix} u^2 \\ u_I^2 \end{Bmatrix} = \begin{bmatrix} M^2 & M_I^2 \end{bmatrix} \begin{Bmatrix} t^2 \\ t_I^2 \end{Bmatrix} \tag{10}$$

The interface conditions are now expressed as

$$t_I = t_I^1 = -t_I^2, \qquad u_I = u_I^1 = u_I^2 \tag{11}$$

Combining Eqs. (9–11), one obtains the final relations that combine the FEM and the BEM relations,

$$\begin{bmatrix} F^1 & F_I^1 & -G_I^1 & 0 \\ 0 & K_I^2 & M_I^2 & K^2 \end{bmatrix} \begin{Bmatrix} u^1 \\ u_I \\ t_I \\ u^2 \end{Bmatrix} = \begin{bmatrix} G^1 & 0 \\ 0 & M^2 \end{bmatrix} \begin{Bmatrix} t^1 \\ t^2 \end{Bmatrix} \tag{12}$$

It is noted that Eq. (12) has the same form as Eq. (8) and may be symbolically expressed as Eq. (8) in a condensed form. The following discussions are based on the matrix Eq. (8) written for the BEM discretization but are valid also for the combined FEM/BEM discretization.

To account for the boundary contact relations, the degrees of freedom corresponding to each Γ_c^k surface must be transformed to the coordinate system corresponding to the surface that results upon the contact occurring between the neighboring bodies. Since this resulting contact surface is not known a priori, the standard approach[11] involves designating the contact surface of one of the elastic bodies as the "master" surface, which is then used for the definition of the coordinate transformations. An alterative approach may be to employ a weighted average of the contacting geometries.[9] The former approach, which is adequate for the small deformation analyses considered here, has been used in this study. The master surface was chosen to be the one corresponding to the unloaded body that had a higher modulus of elasticity. The transformations may be expressed as

$$\begin{gathered} \bar{F}_{ij} = T_i^T F_{ij} T_j \\ \bar{G}_{ij} = T_i^T G_{ij} T_j \\ T_i = \begin{bmatrix} \cos\alpha_i & -\sin\alpha_i \\ \sin\alpha_i & \cos\alpha_i \end{bmatrix} \end{gathered} \tag{13}$$

where T_i denotes the transformation matrix; an overbar denotes a transformed system matrix; however, for simplicity, the overbar is not shown in the subsequent discussions; i is the node being transformed; superscript T denotes matrix transpose; and α_i is the angle of the outward normal to the boundary at node i. For subsequent formulations, it is necessary to express the relations on the contact boundary solely in terms of the unknowns at this boundary. For this purpose, Eq. (8), after appropriate transformations given by Eq. (13) for nodes lying on the contact surface, is first symbolically partitioned as

$$\begin{bmatrix} F_{uu} & F_{ut} & F_{u\tau} & F_{un} \\ F_{tu} & F_{tt} & F_{t\tau} & F_{tn} \\ F_{\tau u} & F_{\tau t} & F_{\tau\tau} & F_{\tau n} \\ F_{nu} & F_{nt} & F_{n\tau} & F_{nn} \end{bmatrix}^k \begin{Bmatrix} u_u \\ u_t \\ u_c^\tau \\ u_c^n \end{Bmatrix}^k = \begin{bmatrix} G_{uu} & G_{ut} & G_{u\tau} & G_{un} \\ G_{tu} & G_{tt} & G_{t\tau} & G_{tn} \\ G_{\tau u} & G_{\tau t} & G_{\tau\tau} & G_{\tau n} \\ G_{nu} & G_{nt} & G_{n\tau} & G_{nn} \end{bmatrix}^k \begin{Bmatrix} t_u \\ t_t \\ t_c^\tau \\ t_c^n \end{Bmatrix}^k + \begin{Bmatrix} f_u \\ f_t \\ f_c^\tau \\ f_c^n \end{Bmatrix}^k \tag{14}$$

where the subscripts u, t, and c refer to the quantities corresponding to the boundaries Γ_u, Γ_t, and Γ_c, respectively; the superscripts τ and n refer, respectively, to the quantities along the tangent and the normal to the contact surface; $t_c^\tau = t^\tau(x), x \in \Gamma_c$; and similarly for t_c^n, u_c^τ, and u_c^n. Substitution of the explicit boundary conditions (2), (3), and (6) leads to

$$\begin{bmatrix} A_{aa} & A_{ac} \\ A_{ca} & A_{cc} \end{bmatrix}^k \begin{Bmatrix} X_a \\ u_c^n \end{Bmatrix}^k = \begin{bmatrix} B_{aa} & B_{ac} \\ B_{ca} & B_{cc} \end{bmatrix}^k \begin{Bmatrix} b_a \\ t_c^n \end{Bmatrix}^k + \begin{Bmatrix} f_a \\ f_c^n \end{Bmatrix}^k \tag{15}$$

with

$$X_a^k = \begin{Bmatrix} t_u \\ u_t \\ u_c^\tau \end{Bmatrix}^k, \qquad b_a^k = \begin{Bmatrix} u_u \\ t_t \\ t_c^\tau \end{Bmatrix}^k, \qquad f_a^k = \begin{Bmatrix} f_u \\ f_t \\ f_c^\tau \end{Bmatrix}^k$$

Eliminating X_a from Eq. (2) and transposing the contact displacement and traction variables to the left-hand side leads to

$$A_1^k x_1^k = b_1^k \tag{16}$$

$$A_1^k = \left[A_{cc} - A_{ca}A_{aa}^{-1}A_{ac}, A_{ca}A_{aa}^{-1}B_{ac} - B_{cc}\right]^k$$

$$x_1^k = \begin{Bmatrix} u_c^n \\ t_c^n \end{Bmatrix}^k$$

$$b_1^k = \left[B_{ca} - A_{ca}A_{aa}^{-1}B_{aa}\right]^k b_a^k + \left(f_c^n - A_{ca}A_{aa}^{-1}f_a\right)^k$$

The resulting system in Eq. (16) is rectangular and undetermined. The number of unknown variables is twice the number of available equations. The remaining conditions to be considered are those of normal nonpenetration on Γ_c. They may be written in the form of complementarity conditions as

$$t^n \delta^n = 0 \tag{17}$$

$$t^n \geq 0 \tag{18}$$

$$\delta^n \geq 0 \tag{19}$$

The complete solution of the contact problem requires the simultaneous consideration of Eqs. (16–19). This may be accomplished by casting these equations as a standard quadratic programming (QP) problem in the optimization theory. The direct substitution of Eq. (16) into Eq. (19) would lead to problems related to the lack of definiteness of the resulting matrix.[14] The contact conditions in Eqs. (17–19) are rewritten as a minimization of an average weighted form to alleviate this problem as

$$\min F = \int_{\Gamma_c} t^n \delta^n \, \mathrm{d}\Gamma_c \tag{20}$$

$$\begin{aligned} t^n &\geq 0 \\ \delta^n &\geq 0 \end{aligned} \tag{21}$$

The constrained minimization problem in Eqs. (20) and (21) is equivalent to the reciprocal variational formulation developed in Refs. 2, 12, and 22. The global minimum of the functional F then corresponds to the solution of the contact problem. The main advantages of enforcing the contact complementarity conditions in the integral form are 1) the integral form has a physical meaning and it corresponds to the minimum of the functional defined on the contact boundary,[12,13] 2) contact conditions are enforced in an average sense that is consistent with the variational framework, and 3) the alternative strategies that use the pointwise form of the objective function in Eq. (20) do not take into account the element type, which is an essential requirement when using higher order elements.

B. QP Formulation

Assuming a linear, elastic material behavior and small displacements, the displacement of the contact surface may be obtained as a superposition of the solutions for 1) the external influences applied on an object without consideration of the contact surface conditions and 2) the contact tractions acting on the body without the presence of external loads. The displacements $\Delta_1^{k^n}$ of the contact boundary Γ_c^k, without the contact restrictions, were given for each body in contact as

$$\Delta_1^{k^n} = u_c^{k^n}\big|_{t_c^{k^n}=0}; \qquad k = 1, 2 \tag{22}$$

or in the expanded form, using Eq. (16), as

$$\Delta_1^{k^n} = \left(A_{cc} - A_{ca}A_{aa}^{-1}A_{ac}\right)^{-1} \times \left[\left(B_{ca} - A_{ca}A_{aa}^{-1}B_{aa}\right)b_a + \left(f_c^n - A_{ca}A_{aa}^{-1}f_a\right)\right]^k$$

The displacements $\Delta_2^{k^n}$ of the contact boundary Γ_c^k due to the contact tractions and in the absence of the external load are written as

$$\Delta_2^{k^n} = u_c^{k^n}\big|_{t_t^k=0} = A_3^k t_c^{k^n}; \qquad k = 1, 2 \tag{23}$$

where

$$A_3^k = \left(A_{cc} - A_{ca}A_{aa}^{-1}A_{ac}\right)^{-1}\left(B_{cc} - A_{ca}A_{aa}^{-1}B_{ac}\right)^k$$

The column i of the operator matrix A_3^k may be interpreted as the displacement of the contact surface Γ_c^k due to the loading given as

$$t_{ci}^{k^n} = 1; \qquad t_{cj}^{k^n} = 0; \qquad j \neq i \tag{24}$$

where $t_{ci}^{k^n}$ denotes the value of the traction at node i. The matrix A_3^k can, therefore, be alternatively obtained by computing the displacements of the contact surface, Γ_c^k, due to the successive applications of load distributions described in Eq. (24) for each $i \in \Gamma_c^k$.

The resulting gap, using superposition, may be written as

$$\delta^n = \epsilon^n + \Delta_1^{1^n} + \Delta_2^{1^n} - \Delta_1^{2^n} - \Delta_2^{2^n} \tag{25}$$

The contact tractions must satisfy the compatibility conditions given as

$$t_c^{2^n} = -t_c^{1^n} \tag{26}$$

The QP form for the contact problem is now obtained by substituting the expression for the resulting gap given in Eq. (25) into the average weighted form of the complementarity condition given by Eqs. (20) and (21). Additionally, to obtain the matrix form for the QP problem, a discretization of the boundary variables u^n, t^n, and ϵ^n is performed using the standard shape functions as

$$u^n = \sum_{i=1}^{N_n} h_i(\xi) u_i^n, \text{ etc.} \tag{27}$$

where N_n is the number of nodes per element, h_i are the shape functions, the superscript i denotes the value of the variable at node i, and ξ is a nondimensional coordinate along the length of the element. The final relation upon these substitutions and discretizations is obtained as

$$\text{Minimize } F_3 = c_3^T x_3 + \tfrac{1}{2} x_3^T H_3^{\text{sym}} x_3 \tag{28}$$

subjected to

$$\left(A_3^1 + A_3^2\right) x_3 \geq b_3 \tag{29}$$

$$x_3 \geq 0 \tag{30}$$

where

$$x_3 = t_c^{1^n}$$

$$c_3 = \bar{J}\left(\epsilon^n + \Delta_1^{1^n} - \Delta_1^{2^n}\right)$$

$$H_3 = \left(A_3^1 + A_3^2\right)\bar{J}$$

$$b_3 = -\left(\epsilon^n + \Delta_1^{1^n} - \Delta_1^{2^n}\right)$$

$$\bar{J} = \sum_{e=1}^{N_e} \bar{J}^e; \qquad \bar{J}_{ij}^e = \int_{-1}^{+1} h_i h_j J \, \mathrm{d}\xi;$$

$$J = \sqrt{\left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \xi}\right)^2}$$

where x_3 is a vector of unknown contact tractions, c_3 denotes the combined integral influence of the initial gap and the external loading, and H_3^{sym} is the symmetric part of the integral of contact displacements due to the contact tractions. The advantage of the present approach lies in the fact that the resulting Hessian matrix is positive definite. This positive definiteness results from the physical interpretation of the quadratic term $x_3^T H_3^{\text{sym}} x_3$ in Eq. (28), which is the discretized form of the portion of the integral in the Eq. (20) given as

$$\int_{\Gamma_c} t_c^n \Delta_2 \, \mathrm{d}\Gamma_c \tag{31}$$

The quadratic term in Eq. (28) represents the work done on the elastic body by the contact tractions only. The contributions due to the external loads are not included in this expression. Since this work done is always positive, the resulting quadratic term is positive definite, and thus the function F_3 is convex. If the complementarity conditions were not imposed in the integral form given in Eq. (20), such an inference could not be drawn. The contact problem is cast in the form of a convex QP problem for which the existence and uniqueness of the solution are defined, and the convergence of the solution is guaranteed in a finite number of steps. The QP formulation of the frictionless contact problem does not require an incremental loading approach to control the contact constraints, and this has shown[26] to be more efficient than the standard trial-and-error methods. It is noted that the contact problem is stated in terms of the contact tractions only. This may significantly reduce the number of variables that have to be considered in the analysis. It is also noted that the operators A_3^k do not depend on the loading and the gap configurations. The method is, therefore, particularly suitable for those contact analyses where a large number of loading and spatial arrangements are required to be considered. The information describing each loading and gap configuration is given by the displacement vectors $\Delta_1^{k^n}$ and the initial gap vector ϵ^n, respectively.

III. Numerical Examples

The proposed formulation was applied for the solution of a number of two-dimensional example problems involving the contact of two or more elastic bodies. In all examples reported here, the contacting objects were modeled using three-noded quadratic boundary elements.[24] The number of boundary elements used for discretizing a boundary segment are shown in corresponding square boxes in the figures that describe the geometry of the problems considered later. Those boundary elements are equally spaced unless mentioned otherwise for a particular boundary segment. For cases involving the coupling of the FEM and BEM discretizations, the regions away from the possible occurrence of contact were modeled using a FEM discretization, whereas the probable contact areas were modeled using a BEM discretization. The FEM regions were modeled using eight-noded serendipity elements. The contact between the two elastic bodies, as stated earlier, was assumed to occur in a node-to-node fashion.

The subroutine QPSOL[27] was used for the numerical solution of the QP problems. For each of the examples, considered here, the null vector and a large constant value vector, respectively, were used as initial guesses for the solution vector, and the tolerances for the maximum permissible violations of constraints were set to the machine precision of 10^{-15}. The null vector is obviously not in a feasible region except in the case when the loading does not activate contact between bodies. Inequalities in Eq. (29) are not satisfied, and an initial feasible point has to be determined by the LP phase of QPSOL. This point is found by increasing the contact traction at a point on the contact surface until the constraints in Eq. (29) are satisfied. Physically, this corresponds to a point support at the contact surface that is gradually removed in QP phase to bring other points in contact. An initial feasible point may alternatively be found by setting the contact tractions to a constant value that is large enough to eliminate contact for given loading. This value can usually be easily estimated in most contact configurations. When contacting surfaces are smooth, such a feasible point is closer to the minimum than the point determined by the LP phase of QPSOL. In all the example problems considered, an average of 30% reduction in the number of QP iterations was observed for a constant value initial vector. All of the results presented in this paper were verified by applying the contact tractions on the undeformed contacting bodies that were obtained from the contact analysis. The displacement compatibility of the resulting configuration was checked to insure that the contact boundary conditions were satisfied.

A. Cantilever Beam in Contact with an Elastic Block

The frictionless contact of a cantilever beam with an elastic block was analyzed assuming a state of plane stress. The respective geometries of the two elastic bodies and the applied loading are shown in Fig. 2. The geometric data used for this problem were

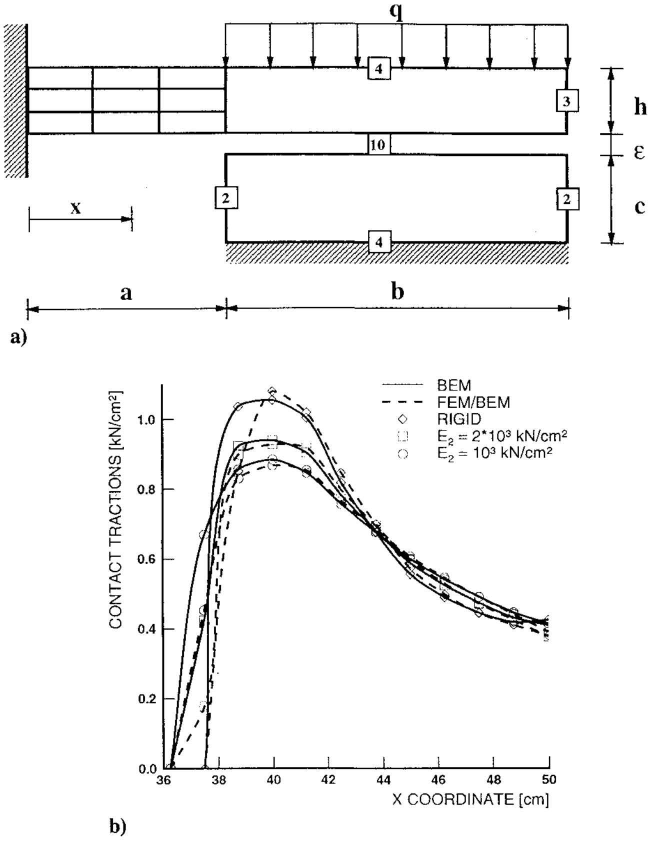

Fig. 2 Cantilever beam in contact with an elastic block.

$a = 20$ cm, $b = 30$ cm, $c = 6.5$ cm, $h = 5$ cm, and the initial gap $\epsilon = 1$ cm; the loading data were $q = 0.4$ kN/cm^2; and the material data were modulus of elasticity $E_1 = 1000$ kN/cm^2, and the Poisson's ratios $\nu_1 = 0.3$ and $\nu_2 = 0$, where the subscripts 1 and 2 refer to the cantilever beam and the supporting block, respectively. A number of cases were analyzed in which the modulus of elasticity E_2 of the supporting block was successively increased as 1000, 2000, 10000 kN/cm^2, and ∞. The beam was modeled using a FEM discretization in its left portion and a BEM discretization in its right portion as shown in Fig. 2. The supporting block was discretized in each case using boundary elements. The problem was also analyzed using a BEM-only discretization for the beam to verify and compare the results obtained using the two discretizations. The FEM discretization used is shown in the mesh in Fig. 2. Since the entries in the corresponding FEM and BEM matrices are of different orders of magnitude, a scaling of the terms in the expression given by Eq. (12) is necessary to obtain good results. The scaling of the corresponding F and G matrices of the BEM is also similarly desirable. The contact tractions along the contact surface for each of the respective cases, and for different values of E_2 considered here, are plotted in Fig. 2. It may be seen that a lower modulus of elasticity produces lower overall tractions and extends the contact region as expected. Also, a good agreement of the FEM/BEM discretization and the BEM-only discretization is obtained.

B. Concentric Cylinders under External Pressure

Two concentric cylinders under the action of an external pressure, $q = 60$ kN/cm^2, applied to the outer cylinder were analyzed. The geometry and the loading considered for this case are shown in Fig. 3. A state of plane strain was assumed. The geometric data used for this problem were $R = 15$ cm, $r = 8.5$ cm, and $\epsilon = 0.5$ cm; the material data used were $E_1 = 1000$ kN/cm^2, and $\nu_1 = \nu_2 = 0$, where the indices 1 and 2 denote the inner and the outer cylinder, respectively. The modulus of elasticity E_2 of the inner cylinder was varied, and its values were taken as 1000, 2000, 10000 kN/cm^2, and ∞, respectively. The problem configuration explained here is expected to produce a uniform distribution of the contact tractions

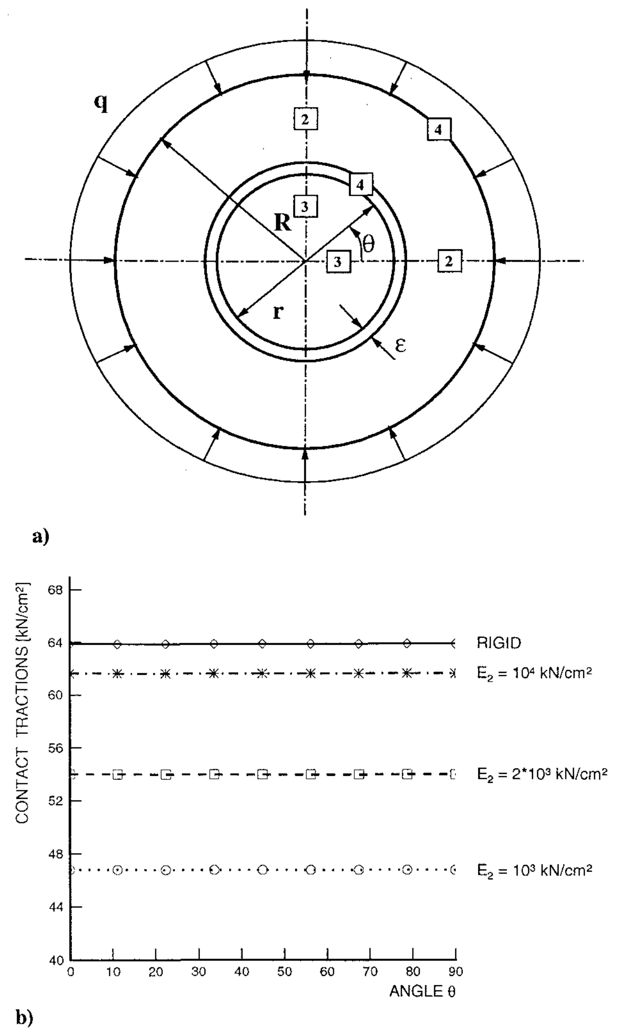

Fig. 3 Concentric cylinders under external pressure.

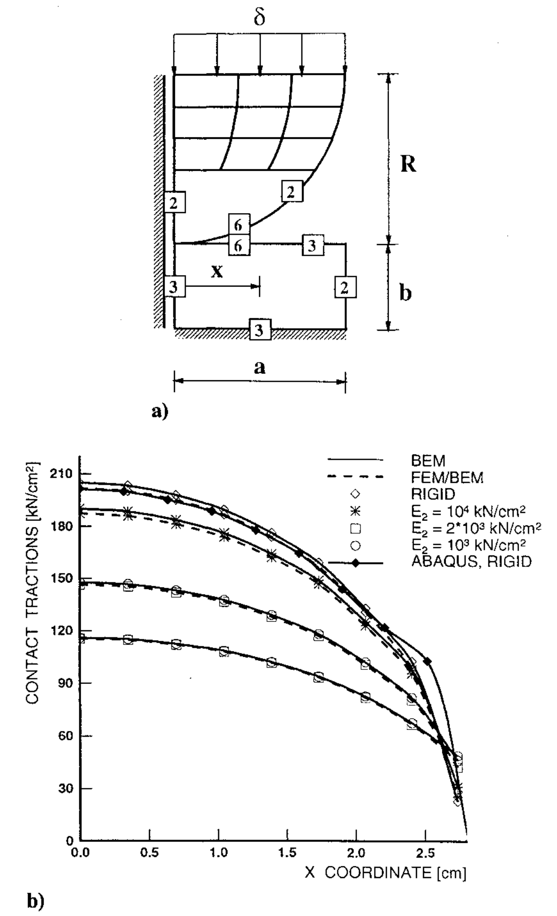

Fig. 4 Infinite elastic half-cylinder on an elastic foundation.

on the contact surface. This result serves to validate the present formulations. Because of the double symmetry of the problem, only a quarter of the model was analyzed. The distributions of contact tractions along the cylinder angle, for each of the respective cases corresponding to the different values of E_2 considered, are shown in Fig. 3. As expected, a uniform traction distribution was obtained in each case, thus validating the present developments. It is also noted that the results for the rigid case serve as an upper limit on the tractions developed at the contact surface.

C. Infinite Elastic Half-Cylinder on an Elastic Foundation

An infinitely long half-cylinder indenting into an elastic foundation is studied next. A state of plane strain was assumed for this analysis. The top face of the half-cylinder was subjected to a uniform unit displacement δ. The material data used for this problem were $E_1 = 1000$ kN/cm^2, $\nu_1 = 0.3$, and $\nu_2 = 0$, where the indices 1 and 2 denote the half-cylinder and the supporting block, respectively. A number of cases for which the modulus of elasticity E_2 of the block was successively varied as 1000, 2000, 10000 kN/cm^2, and ∞, respectively, were studied. The geometry and the displacement loading were defined by the following parameters: $R = 8$ cm, $a = 8$ cm, $b = 4$ cm, and $\delta = 1$ cm. Because of the double symmetry of the problem, only a quarter of the geometry was modeled as shown in Fig. 4. The contactor (half-cylinder) was modeled using a FEM/BEM discretization with the respective discretization domains as shown in Fig. 4. The problem was also analyzed using a BEM-only discretization. The case of $E_2 = \infty$ has also been analyzed using the commercial FEM package ABAQUS,[28] for the purpose of comparison of results. The null vector initial guess resulted in 1 LP and 17 QP iterations, and the constant value vector, $x_i = 400$, required 13 QP iterations. The traction distributions along the contact surface for the different moduli of elasticity of the foundation block considered here, as well as for the different discretizations described earlier, are shown in Fig. 4. A good correlation of results was obtained for the respective FEM/BEM, FEM-only, and BEM-only discretizations. The tendency of a lower modulus of elasticity to lower the contact tractions was also observed in the present cases.

D. Elastic Bolt in a Thick Plate

The contact of a thick rectangular plate with an enclosed elastic bolt was considered. A uniform displacement δ was applied on the right vertical face of the plate. One-half of this configuration was modeled due to the symmetry of the geometry and the applied loading as shown in Fig. 5. A state of plane strain was assumed. The material properties used for this problem were $E_1 = 1000$ kN/cm^2, $\nu_1 = 0.3$, and $\nu_2 = 0$, where the indices 1 and 2 denote the plate and the bolt, respectively. The geometry and the loading parameters were defined as $a = 6$ cm, $b = 1.4$ cm, $c = 0.8$ cm, $h = 6$ cm, $\epsilon = 0.2$ cm, and $\delta = 1$ cm. Three different moduli of elasticity, E_2, of the bolt, of values 1000, 2000, and 10000 kN/cm^2, together with the rigid surface condition, respectively, were used to study a number of cases for this analysis. A rigid core, shown as a shaded region in Fig. 5, was provided to constrain the rigid-body translation of the entire arrangement. The probable contact region was considered to be the left quarter of the outer perimeter of the bolt. The contact tractions along the bolt periphery were computed and are shown in Fig. 5. As in the previous cases, the rigid surface condition provides an upper limit for the contact tractions developed on the surface of the bolt.

E. Rigid Bolts in a Thick Plate

A pair of rigid bolts placed in a thick plate as shown in Fig. 6 were considered. Three cases were studied: 1) the centers of the plate holes coincide with the longitudinal symmetry axis of the plate as shown by a solid line, 2) the hole centers are eccentric by a distance,

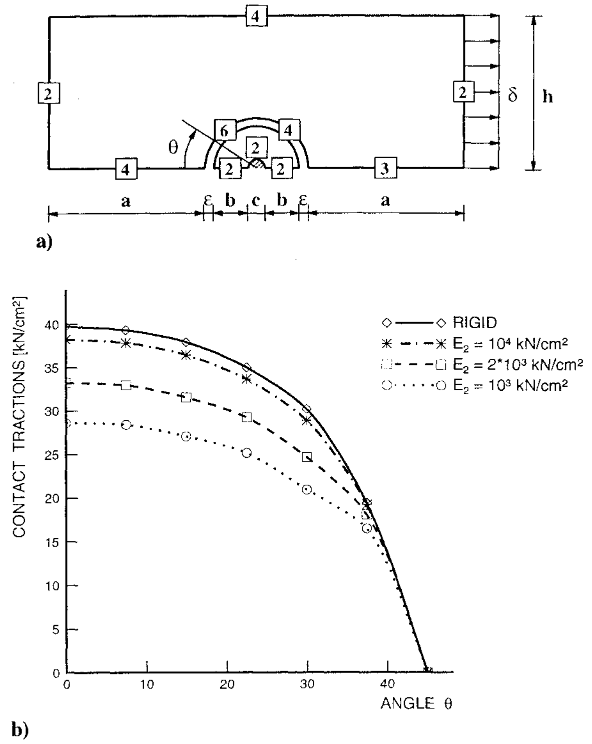

Fig. 5 Elastic bolt in a thick plate.

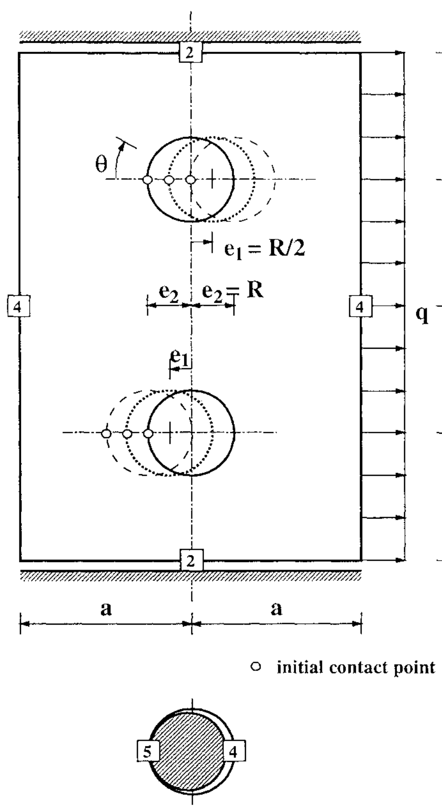

Fig. 6 Rigid bolts in a thick plate.

$e_1 = R/2$, with respect to the longitudinal axials as shown by dotted lines, and 3) the hole centers are eccentric by a distance, $e_2 = R$, with respect to the same axis as shown by dashed lines. A state of plane strain was assumed for this analysis. The numerical data used were $E = 1000$ kN/cm^2, $\nu_1 = 0.3$, $a = 8$ cm, $b = 6$ cm, $R = 2$ cm, and $\epsilon = 0.2$ cm. The distance ϵ denotes the difference between the respective diameters of the plate hold and the bolt. The initial points of contact are denoted by circles in Fig. 6. A uniformly distributed load of intensity, $q = 25$ kN/cm^2, was applied on the right edge

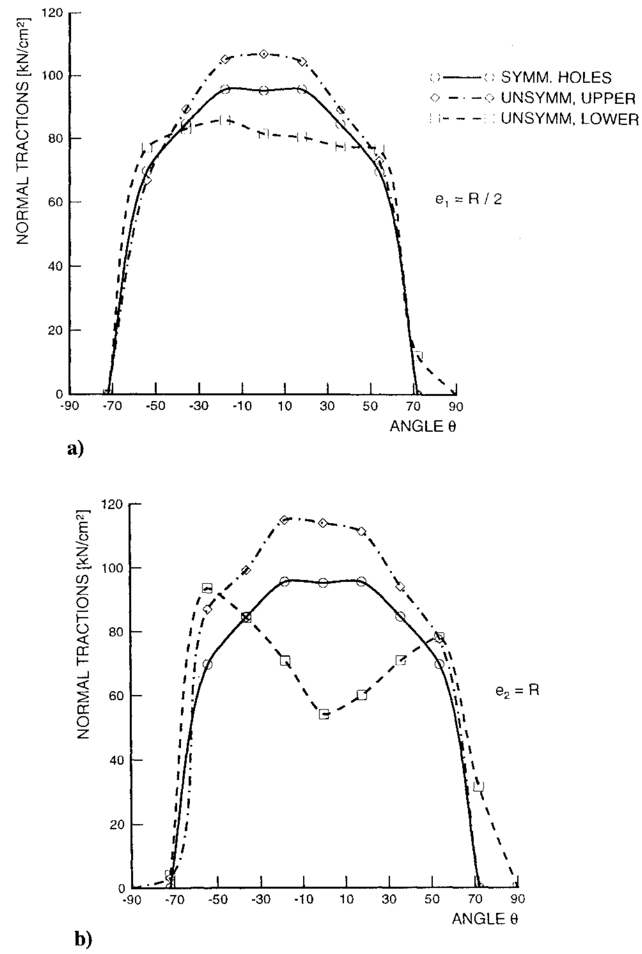

Fig. 7 Effect of hole eccentricity in a bolt connection.

of the plate. The traction distributions along the contact periphery for the eccentricities e_1 and e_2 were plotted in Figs. 7a and 7b, respectively. It is observed that the eccentric configurations result in higher tractions on the upper hole, which is eccentric towards the loaded edge. This example demonstrates the capability of the present developments to effectively treat multiple contact regions.

IV. Conclusions

A quadratic programming formulation based on the contact complementarity conditions has been presented in this paper for the boundary element method. The proposed formulation is given solely in terms of the unknown contact tractions, thus reducing the size of the resulting numerical model. The present method is particularly suitable for the analysis of contact problems where a large number of loading and gap conditions have to be considered. A method for the incorporation of different discretization methods, namely, the FEM and the BEM, for the analysis of contact problems has also been presented. A number of example problems were solved to demonstrate the validity of the present formulation.

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